Lecture 23.

Fubini-Tonelli for completions.

Even when μ, ν are complete, $\mu\times\nu$ need not be. If $(X\times Y, \mathcal{L}, \lambda)$ denotes the completion of $\mu\times\nu$, then FT still holds, provided μ and ν are complete. The proof is indicated (reduced to FT) in a HW exercise.

The Lebesgue Measure on $\mathbb{R}^n$.

Let $(\mathbb{R}, \mathcal{L}, m)$ denote 1-D Lebesgue measure, which is complete. We let m^n, $\mathcal{L}^n$ denote the completion of $m\times\ldots\times m$ and its σ-algebra, respectively. (The completion could also have been done from the product of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$.)

The n-D LM shares many features w/ 1-D. We summarize the most important.

<u>Thm 1</u>. Let $E \in \mathcal{L}^n$. Then:

(i) $m(E) = \inf\{m(U) : E \subseteq U \text{ open}\} = \sup\{m(K) : E \supseteq K \text{ compact}\}$.

(ii) $E = A_1 \cup N_1 = A_2 \setminus N_2$, where N_1, N_2 are null, $A_1 \in F_\sigma$, $A_2 \in G_\delta$.

(iii) If $m(E) < \infty$, then $\forall \varepsilon > 0$ $\exists$ finite, disjoint union $\bigcup_{k=1}^{n} R_k$ of rectangles that are products of intervals s.t.

$$m\Big(E \,\Delta\, \bigcup_{k=1}^{n} R_k\Big) = m\big(E \setminus \cup R_k\big) + m\big(\cup R_k \setminus E\big) < \varepsilon.$$

The proofs essentially reduce to the results in 1-D by using the definition of the outer measure by coverings of sets in $\mathcal{A} = \{$finite disjoint unions of rectangles$\}$. Details are DIY (or see Folland.)

Thm 2. (i) Simple fcns of the form $\varphi = \sum_{j=1}^{n} c_j \chi_{R_j}$, where R_j are products of intervals are dense in $L^1(\mathbb{R}^n, m^n)$.

(ii) Cont. fcns w/ compact support (i.e. that vanish outside a compact set) are dense in $L^1(\mathbb{R}^n, m^n)$.

Pf. (i) Fix $\varepsilon > 0$. Take seq. of simple fcns φ_n s.t. $\varphi_n \to f$ a.e. and $|\varphi_n| \nearrow |f|$.
By DCT, $\int |f - \varphi_n| \to 0 \Rightarrow \exists\, \psi := \varphi_N$ s.t. $\int |f - \psi| < \varepsilon/2$. Now,

$$\psi = \sum_{j=1}^{m} a_j \chi_{E_j}$$

where $m(E_j) < \infty$. For each j, by Thm 1 (iii), pick products of intervals R_{jk} s.t. $m(E_j \triangle \bigcup_{1}^{m_j} R_{jk}) < \delta_j \varepsilon$

to be chosen

Let $\varphi = \sum_{j=1}^{m} a_j \sum_{k=1}^{m_j} \chi_{R_{jk}}$, which is of the desired form. We have

$$\int |f-\varphi| \le \int |f-\psi| + \int |\psi - \varphi|$$

$$\le \frac{\varepsilon}{2} + \sum_{j=1}^{m} |a_j| \underbrace{\int \Big| \chi_{E_j} - \sum_{k=1}^{m_j} \chi_{R_{jk}} \Big|}_{m(E_j \Delta \bigcup_{1}^{m_j} R_{jk})}$$

$$\le \frac{\varepsilon}{2} + \sum_{j=1}^{m} |a_j| \delta_j \varepsilon .$$

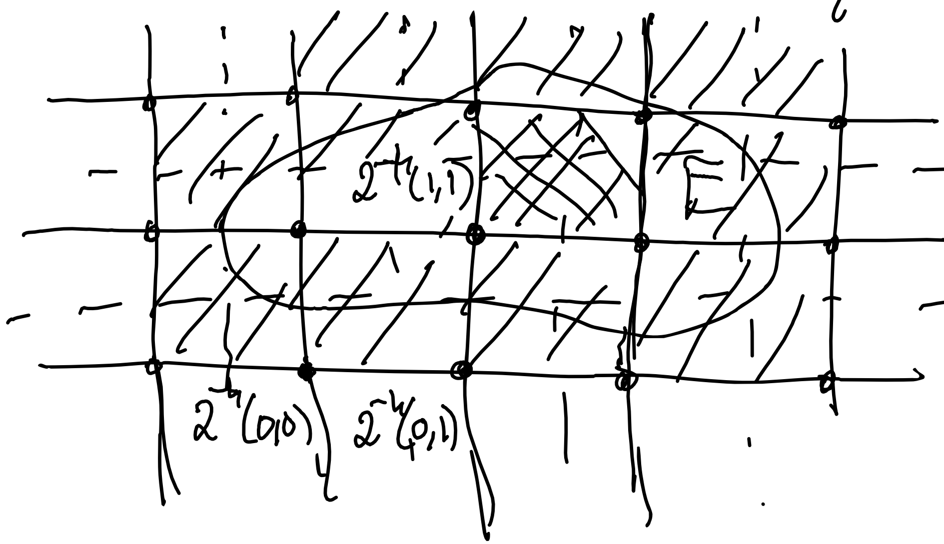
Thus, we pick δ_j s.t. $\le \varepsilon/2$,

This proves (i).

(ii) We just need to approximate the simple functions in (i). Since the sets R_{jk} are products of intervals this can be done as in the 1-D case.

Jordan content.

The notion of Jordan content is addressed in Folland, as a comparison w/ "advanced calculus". We just give a brief discussion here. Let $E \subseteq \mathbb{R}^n$ ($n=2$ in my pics)



Take a dyadic decomposition into "cubes". For $k \in \mathbb{Z}$, consider collection $\mathcal{C}_k$ of cubes (= products of intervals) whose lengths are 2^{-k} and whose endpoints are of the form $m \cdot 2^{-k}$.

We shall assume that E is <u>bounded</u>.

Let $\underline{A}(E,k) = \bigcup \{Q \in \mathcal{C}_k : Q \subseteq E\}$

$\overline{A}(E,k) = \bigcup \{Q \in \mathcal{C}_k : Q \cap E \neq \emptyset\}$.

Clearly $\underline{A}(E,k) \subseteq E \subseteq \overline{A}(E,k)$

and as $k \nearrow$, $\underline{A}(E,k) \nearrow$ and

$\overline{A}(E,k) \searrow$. (See pic on previous page.)

Both are Borel sets, and so

$$m^n(\underline{A}(E,k)) \nearrow \underline{\varkappa}(E) \leftarrow \text{inner content.}$$

$$m^n(\overline{A}(E,k)) \searrow \overline{\varkappa}(E) \leftarrow \text{outer content.}$$

If $\underline{\varkappa}(E) = \overline{\varkappa}(E)$, this is called the <u>Jordan content</u> of E, $\varkappa(E)$.

<u>Prop 1.</u> Jordan content of E exists $\Rightarrow$ $E \in \mathcal{L}^n$ and $m^n(E) = \varkappa(E)$.

Pf. Define $\underline{A}(E) = \bigcup_{k=1}^{\infty} \underline{A}(E,k)$ and $\overline{A}(E) = \bigcap_{k=1}^{\infty} \overline{A}(E,k)$ both of which are Borel sets by construction. We have $m^n(\underline{A}(E)) = \underline{\alpha}(E)$ and $m^n(\overline{A}(E)) = \overline{\alpha}(E)$ by continuity from below and above. Thus,

$$\underline{A}(E) \subseteq E \subseteq \overline{A}(E),$$

and $m^n(\overline{A}(E) \setminus \underline{A}(E)) = \overline{\alpha}(E) - \underline{\alpha}(E) = 0$.

Thus, $E \setminus \underline{A}(E)$ is contained in the nullset $\overline{A}(E) \setminus \underline{A}(E)$ and, hence since m^n is complete, $E \setminus \underline{A}(E)$ is measurable $\Rightarrow$ $E = \underline{A}(E) \cup (E \setminus \underline{A}(E))$ is measurable. $\boxtimes$